CONSUMPTION GROWTH 101

RUNNING THE NUMBERS

There once was a man from Trinity
Who computed the square root of infinity
But the number of digits
Gave him the fidgets
So he dropped math and took up divinity
Anon

This tutorial covers various calculations relating to exponential growth. To perform these calculations you will need either a scientific calculator or know how to use mathematical functions in a spreadsheet program such as Excel.

Exponential Growth

If a value is growing at an annual percentage growth rate of *r* then after *t* years the value will be a multiple of the present value equal to:

$$e^{rt}$$

r is a decimal fraction. If the growth rate is 2 percent then *r* is

$$2 \div 100 = 0.02$$

Example

A state has one million registered vehicles on the road. The number of registered vehicles is growing by 1 percent per annum. At that rate how many registered vehicles will there be in 15 years time?

Solution

The multiple of the present number of vehicles will be e^{rt} where

$r = 0.01$
$t = 15.$

$e^{(0.01 \times 15)}$

$= e^{0.15}$

Enter 0.15 into your calculator and press e^x.

Alternatively, enter the following equation into an Excel spreadsheet cell.

=EXP(0.15)

The result should be 1.161834...

Thus, at a 1 percent growth rate in 15 years time the number of registered vehicles will be

1.161834 x 1,000,000 = 1,161,834 vehicles

Larger Numbers

What if the numbers are larger? Let's say that instead of 15 years the value of *t* is 3 million years?

If *r* is 1 percent then the multiple is $e^{30,000}$. Enter 30,000 into a calculator and press e^x or enter =EXP(30000) into an Excel cell.

The calculator will report an error and Excel will display #NUM!. The resulting number is so large that it exceeds the maximum range for that device.

There is a neat trick that can be used to get around this problem.

When numbers get very large they are generally represented by shorthand notation.

For example, the 13 digit number

4,398,046,511,104

can be written as

4.398×10^{12}

On a calculator this might appear as 4.398 E12.

The 12 in the above number is called the *exponent*. When numbers get very large it is the size of the exponent that is useful for comparison. If the number were written out in full, the number of digits would be the exponent plus 1.

To find out the base 10 exponent of the result of the calculation e^{rt}, divide rt by 2.3 (or, more precisely: $ln(10)$).

$$e^{rt} = 10^{(rt / 2.3)}$$

If rt = 30,000 then

$$e^{rt} = 10^{(30,000/2.3)}$$

$$= 10^{13043}$$

The number is 13,044 digits long! Picture a 1 with 13,043 zeros after it.

Choosing a smaller number that the calculator should be able to handle, let's say that rt equals 150.

Calculate e^{150} and the result will be represented as something like

1.39371 E65.

Divide 150 by 2.3 and, lo and behold, the answer is 65.

A common favourite for the comparison of large numbers is the estimate of the number of atoms in the universe: 10^{80}. Hence for most purposes the fact that a typical calculator can only handle numbers up to 10^{99} doesn't really matter.

Calculating Average Growth Rate

What if the multiplier and duration are known; how can the average growth rate be calculated? To work this out requires the use of another function the natural logarithm. This function is usually labelled ln or log_e or LN.

The average percentage growth rate is

$$100 \times ln\text{ (multiplier) / duration}$$

Example

For arguments sake, let's say that the duration of human history from two people to the present population of 6.6 billion is 6,000 years. What would be the average population growth rate over that time?

$$r = 100 \times ln(3{,}300{,}000{,}000) / 6{,}000$$

$$= 0.365\%$$

If the duration is 60,000 years then clearly the average becomes 0.0365% and so on.

The Rule of 70

If a value is growing at a fixed rate it will repeatedly double in a fixed interval of time. That interval can be approximated by dividing 70 by the rate in percent.

$$\text{Doubling period} = 70 \div \text{growth rate in } \%$$

For example: If a value is growing at 2 percent per annum, the value will double in 35 years. It will double again in the next 35 years and will therefore be four times the value at the start. If the rate of growth is 3 percent then the doubling period will be $70 \div 3 = 23.3$ years.

The rule of 70 is easy to remember and calculate. The more precise calculation is

$$ln(2) / r$$

Doubling of Consumption Equals All Previous Consumption

Take a value which is growing at a constant rate and has always been growing at the same rate. If the value which is growing is a flow, such as consumption, then the total amount consumed in a doubling period equals all of the consumption for all previous periods of doubling. There is no calculation to learn with this one. Just take it that it is so. (The proof is quite straightforward with a basic understanding of calculus.)

In the real world growth rates are never constant. But the principle is still useful. If the average rate of growth in the earlier periods was greater than at present, then the length of time in which the total consumption will exceed all previous consumption will be shorter than the doubling period. If the average growth in earlier periods was less, then the period to equal all previous cumulative consumption will be longer than the consumption doubling period.

Let's say that since the first electricity generators were built, historical electricity consumption growth has varied over time but averaged around 4 or 5 percent per annum. If consumption growth is currently steady at 2 percent then the present doubling period is 35 years. In some length of time less than 35 years the total amount of electrical energy consumed between now and then will exceed all that which has been consumed for all previous time before now.

Geometric Growth

If a value doubles three times then the result is 2x2x2 = 8 times the original value. If the number of doubling periods is 42 then the multiple is 2x2x2x2x.... 42 times. There is an easier way of calculating the answer than multiplying 2 in a calculator 42 times. That is by recognising that this is equivalent to 2^{42}.

A scientific calculator will usually have a button which might be labelled a^x. 2^{42} can be calculated by entering

$$2 \ a^x \ 42 =$$

In Excel this can be calculated by entering

=2^42

The result of this calculation is 4,398,046,511,104.

Another example: If a water lily is doubling in size every day, what size will it be in 10 days time?

$2^{10} = 1024$ times the size it is now.

Number of Doubling Periods

What if the multiple is known and we want to know how many doubling periods it equals?

The number of doubling periods of a value X is equal to

$$ln(X) / \ ln(2)$$

If the amount of sunlight reaching the earth in a year is 11,514 times the amount of energy marketed in the world, we can work out how many doubling periods this equals by calculating

$$\ln(11{,}514) / \ln(2)$$

which equals 13.49 doubling periods.

Combining Doubling Periods and The Rule of 70

Having worked out the number of doubling periods, only simple arithmetic is now needed to calculate various scenarios of growth rates. This is applicable when dealing with growth in the rate of consumption of renewable resources. If the rate of energy consumption growth is 1 percent then the doubling period is 70 ÷ 1 = 70 years. To work out how many years at 1 percent growth before energy demand exceeds total incident solar energy, simply multiply 70 by 13.49 to get 944 years. For a 2 percent growth rate multiply 35 years times 13.49 to get 469 years.

With a calculator, the shorter and more precise method for combining doubling periods and the Rule of Seventy is to calculate

$$\ln(X) / r$$

Exponential Consumption of a Finite Resource

Let's say that there is a resource, such as fossil fuel, which can only be used once and is being consumed at an exponentially growing rate *r*. How long will it last?

If the present resource size is expressed as a multiple *n* times present annual consumption then the number of years it will last is

$$T = \ln(rn+1) / r$$

Example

Let's say that the amount of hot rock energy available is estimated as being 7,500 times the present annual electricity consumption of the nation. If electricity consumption were to continue growing at 2 percent per annum, how long would this resource last?

$$T = \ln((0.02 \times 7500) + 1) / 0.02$$

$$= 250.86 \text{ years.}$$

If instead of 2 percent, electricity consumption growth were 1 percent, how long would it last?

$$T = ln((0.01 \times 7500) + 1) / 0.01$$

$$= 433.07 \text{ years.}$$

If the resource is reappraised as being 75,000 times present annual consumption, and growth is 1 percent, what difference will this make?

$$T = ln((0.01 \times 75000) + 1) / 0.01$$

$$= 662.14 \text{ years.}$$

Notice that even though the resource size has increased by a factor of 10, the length of time that it will last has only increased by a factor of around 1.5. It is by performing these types of calculations that one can gain an appreciation of the nature of exponential consumption growth.

Finding Hubbert's Peak

What if, (as in the peak oil concept), the consumption of a particular resource typically grows exponentially to a peak and then falls exponentially. How can some estimate be made of the timing of the peak?

Since the curve is roughly symmetrical the peak will occur when half of the resource has been consumed. As pointed out earlier, the cumulative amount of consumption will double in the same time as the rate of consumption doubles. Therefore, as an approximation, half the resource will have been consumed in one doubling period less than that it would have taken for the entire resource to be consumed if there was no peak.

<u>Example</u>

A non-renewable resource is 7,500 times present annual consumption. Consumption is growing at 2 percent per annum. If consumption continues to grow at 2 percent per annum until a production peak is reached and production growth and consumption growth stop, at what point will this occur?

From the previous section the resource would otherwise have lasted 250 years if production continued to increase until the resource was completely exhausted. Since the doubling period for 2 percent is 35 years, the peak will occur 250 – 35 = 215 years from now.

Cumulative Consumption

What if we wish to perform the reverse calculation: i.e. what multiple of present annual consumption will be cumulatively consumed over a period of exponentially growing consumption? The multiple of present annual consumption is calculated as

$$n = (e^{rt} - 1) / r$$

<u>Example</u>

If energy consumption were growing at 1 percent per annum, what resource of non-renewable energy would be required to meet all the cumulative requirements for the next 1,000 years?

$$n = (e^{(0.01 \text{ x } 1000)} - 1) / 0.01$$

$$= 2{,}202{,}547 \text{ times present annual consumption}$$

Actual energy consumption growth in the early 21st century is typically closer to 2 percent than 1 percent. What multiple of present annual requirements is required if the growth rate is 2 percent?

$$n = (e^{(0.02 \text{ x } 1000)} - 1) / 0.02$$

= 24,258,259,720 times present annual consumption

Yes, twenty-four *billion* times.